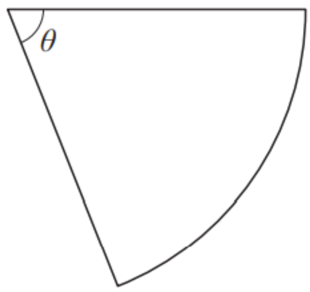
A-level
MATHS
Trigonometry (Topic E)

Total number of marks: 41

3 The diagram below shows a sector of a circle.



$A = \frac{1}{2} r^2 \theta$

$A = \frac{1}{2}(4)^2(0.8)$

$= 6.4$

The radius of the circle is 4 cm and $\theta = 0.8$ radians.

Find the area of the sector.

Circle your answer.

[1 mark]

$1.28\,\text{cm}^2$ $3.2\,\text{cm}^2$ $6.4\,\text{cm}^2$ $12.8\,\text{cm}^2$

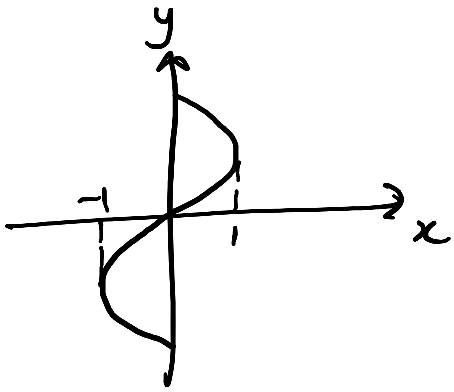
1 $\mathrm{f}(x) = \arcsin x$

State the maximum possible domain of f

Tick (✓) **one** box.

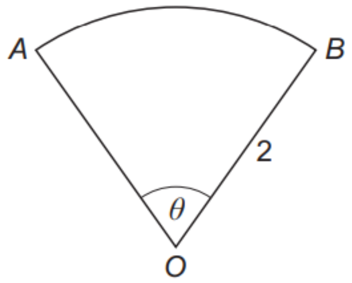
[1 mark]

$\{x \in \mathbb{R} : -1 \le x \le 1\}$	✓
$\left\{x \in \mathbb{R} : -\frac{\pi}{2} \le x \le \frac{\pi}{2}\right\}$	
$\{x \in \mathbb{R} : -\pi \le x \le \pi\}$	
$\{x \in \mathbb{R} : -90 \le x \le 90\}$	



$-1 \le x \le 1$

3 The diagram shows a sector OAB of a circle with centre O and radius 2



$$2\pi r \times \frac{\theta}{2\pi} + 2 + 2 = 6$$
$$r\theta + 4 = 6$$
$$r\theta = 2$$
$$2 \times \theta = 2$$
$$\theta = 1$$

The angle AOB is θ radians and the perimeter of the sector is 6

Find the value of θ

Circle your answer.

[1 mark]

1 $\qquad$ $\sqrt{3}$ $\qquad$ 2 $\qquad$ 3

4 Using small angle approximations, show that for small, non-zero, values of x

$$\frac{x \tan 5x}{\cos 4x - 1} \approx A$$

where A is a constant to be determined.

[4 marks]

$$\frac{x \tan 5x}{\cos 4x - 1}$$

$$\approx \frac{x(5x)}{(1-8x^2)-1} = \frac{5\cancel{x^2}}{-8\cancel{x^2}} = -\frac{5}{8}$$

$$\boxed{A \approx -\frac{5}{8}}$$

$$\tan 5x \approx 5x$$
$$\cos 4x \approx 1 - \frac{(4x)^2}{2}$$
$$= 1 - \frac{16x^2}{2}$$
$$= 1 - 8x^2$$

8 (a) Prove the identity $\dfrac{\sin 2x}{1+\tan^2 x} \equiv 2\sin x \cos^3 x$

[3 marks]

$$\frac{\sin 2x}{1+\tan^2 x} \equiv \frac{\sin(x+x)}{1+\tan^2 x}$$

$$\equiv \frac{2\sin x\cos x}{\sec^2 x}$$

$$\equiv \frac{2\sin x \cos x}{\frac{1}{\cos^2 x}} = \boxed{2\sin x\cos^3 x}$$

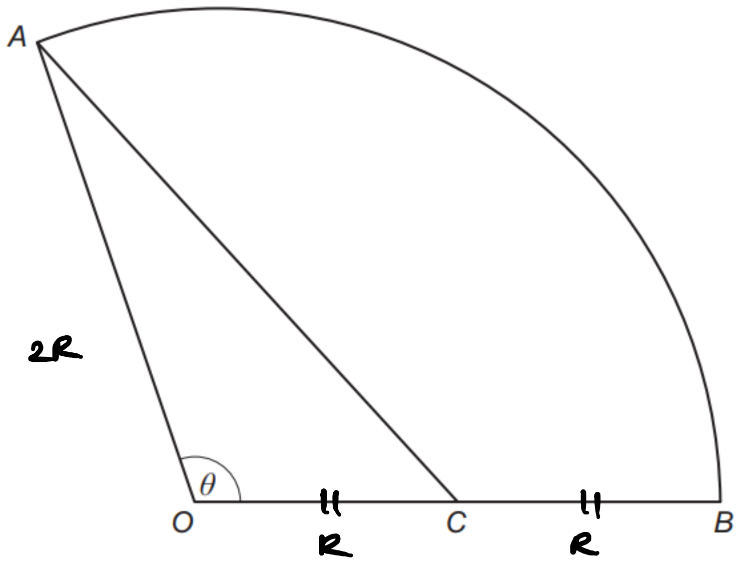
$$\tan^2 x + 1 = \sec^2 x$$

$$\sec^2 x = \frac{1}{\cos^2 x}$$

8 The diagram shows a sector of a circle OAB.

C is the midpoint of OB.

Angle AOB is θ radians.



8 (a) Given that the area of the triangle OAC is equal to one quarter of the area of the sector OAB, show that $\theta = 2\sin\theta$

[4 marks]

$OAC \times 4 = OAB$

$\left[\frac{1}{2}(R)(2R)\sin\theta\right] \times 4 = \pi(2R)^2 \times \frac{\theta}{2\pi}$

$4R^2\sin\theta = 2R^2\theta$

$2\sin\theta = \theta$

$\boxed{\theta = 2\sin\theta}$

12 (a) Show that the equation

$$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$$

can be written in the form

$$a\operatorname{cosec}^2 x + b\operatorname{cosec} x + c = 0$$

[2 marks]

$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$ $\quad 1 + \cot^2 x = \operatorname{cosec}^2 x$

$2(\operatorname{cosec}^2 - 1) + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$ $\quad \cot^2 x = \operatorname{cosec}^2 x - 1$

$2\operatorname{cosec}^2 x - 2 + 2\operatorname{cosec}^2 x - 1 - 4\operatorname{cosec} x = 0$

$\boxed{4\operatorname{cosec}^2 x - 4\operatorname{cosec} x - 3 = 0}$

12 (b) Hence, given x is obtuse and

$$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$$

find the exact value of $\tan x$

Fully justify your answer.

[5 marks]

$4\operatorname{cosec}^2 x - 4\operatorname{cosec} x - 3 = 0$

$2x \quad -3 \quad = -6$

$2x \quad 1 \quad = 2$

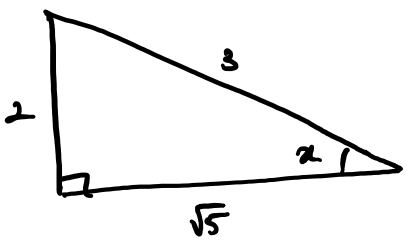
$(2\operatorname{cosec} x - 3)(2\operatorname{cosec} x + 1) = 0$

$\operatorname{cosec} x = \frac{3}{2}$ or $-\frac{1}{2}$ $\qquad \operatorname{cosec} x = \frac{1}{\sin x}$

$\frac{1}{\sin x} = \frac{3}{2}$ or $-\frac{1}{2}$

$\sin x = \frac{2}{3}$ or -2

impossible



$3^2 = 2^2 + ?^2$

$9 = 4 + ?^2$

$? = \sqrt{5}$

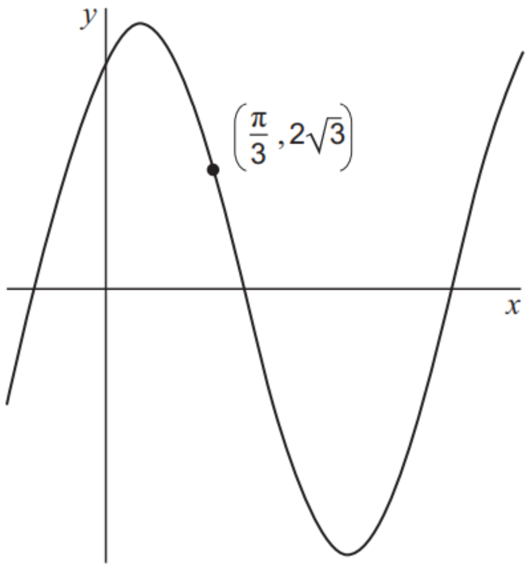
$\boxed{\tan x = \frac{2}{\sqrt{5}}}$

6 A curve has equation

$$y = a\sin x + b\cos x$$

where a and b are constants.

The maximum value of y is 4 and the curve passes through the point $\left(\frac{\pi}{3}, 2\sqrt{3}\right)$ as shown in the diagram.



Find the exact values of a and b.

[6 marks]

$y = a\sin x + b\cos x = R\cos(x+\alpha)$

$= R\cos x\cos\alpha + R\sin x\sin\alpha$

$= R\sin\alpha\sin x + R\cos\alpha\cos x$

$2\sqrt{3} = a\sin\left(\frac{\pi}{3}\right) + b\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}a + \frac{1}{2}b$

$\sqrt{3}a + b = 4\sqrt{3}$

$b = 4\sqrt{3} - \sqrt{3}a$

$\left(\frac{\pi}{3}, 2\sqrt{3}\right)$

$(R\sin\alpha)^2 + (R\cos\alpha)^2 = a^2 + b^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = a^2 + b^2$

$R^2 = a^2 + b^2$

R = maximum y-value so 4

$4^2 = a^2 + b^2 = 16$

$a^2 + (4\sqrt{3} - \sqrt{3}a)^2 = 16$

$a^2 + 48 - 24a + 3a^2 - 16 = 0$

$4a^2 - 24a + 32 = 0$

$a^2 - 6a + 8 = 0$

$(a-4)(a-2) = 0$

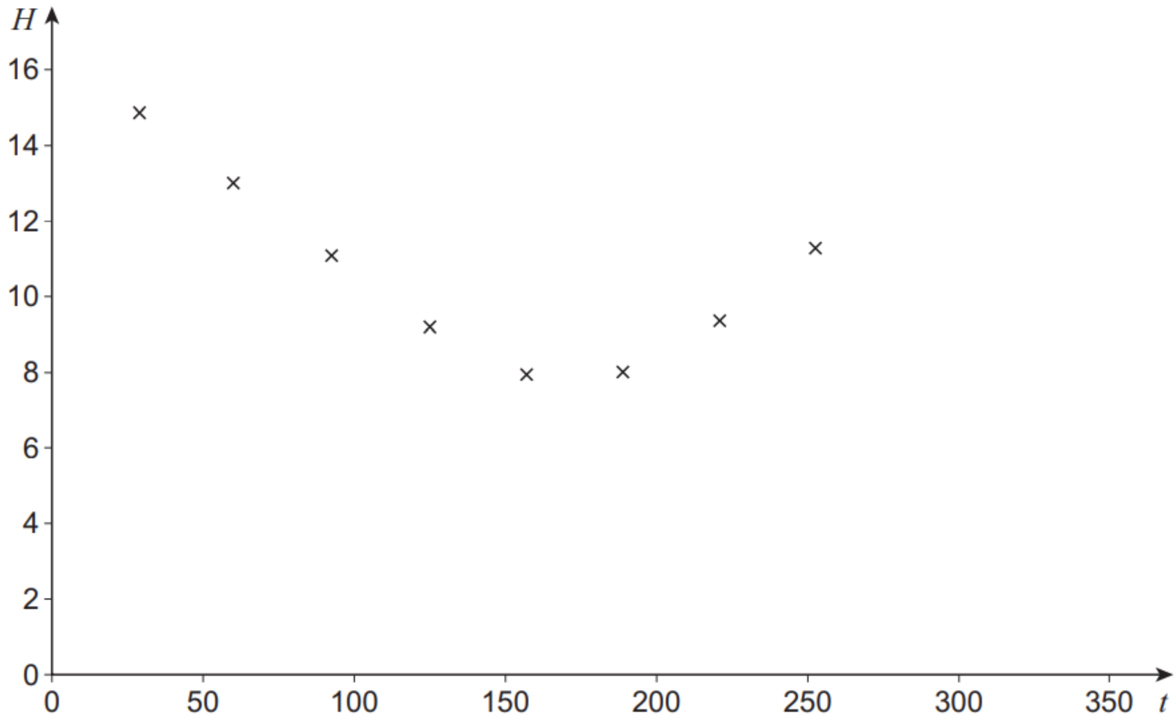
$\boxed{a = 4 \text{ or } 2 \quad b = 0 \text{ or } 2\sqrt{3}}$

8 Mike, an amateur astronomer who lives in the South of England, wants to know how the number of hours of darkness changes through the year.

On various days between February and September he records the length of time, H hours, of darkness along with t, the number of days after 1 January.

His results are shown in **Figure 1** below.

Figure 1



Mike models this data using the equation

$$H = 3.87\sin\left(\frac{2\pi(t+101.75)}{365}\right) + 11.7$$

8 (a) Find the minimum number of hours of darkness predicted by Mike's model. Give your answer to the nearest minute.

[2 marks]

when $\sin\left(\frac{2\pi(t+101.75)}{365}\right) = -1$

$H = 3.87(-1) + 11.7 = 7.83$ hours

$\Downarrow$

7 hours 50 minutes

$\boxed{470 \text{ minutes}}$

8 (b) Find the maximum number of consecutive days where the number of hours of darkness predicted by Mike's model exceeds 14

[3 marks]

$14 < 3.87\sin\left(\frac{2\pi(t+101.75)}{365}\right) + 11.7$

$2.3 < 3.87\sin\left(\frac{2\pi(t+101.75)}{365}\right)$

$\sin\left(\frac{2\pi(t+101.75)}{365}\right) > \frac{230}{387}$

$\sin\left(\frac{2\pi(t+101.75)}{365}\right) = \frac{230}{387}$

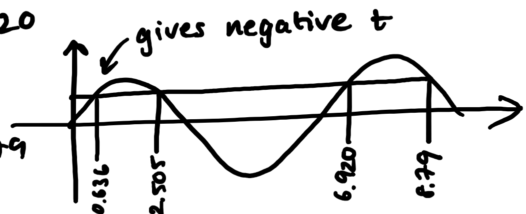
$\frac{2\pi(t+101.75)}{365} = 0.636 \rightarrow 6.920$

$2.505 \rightarrow 8.79$

$t = 300, 409$

$409 - 300$

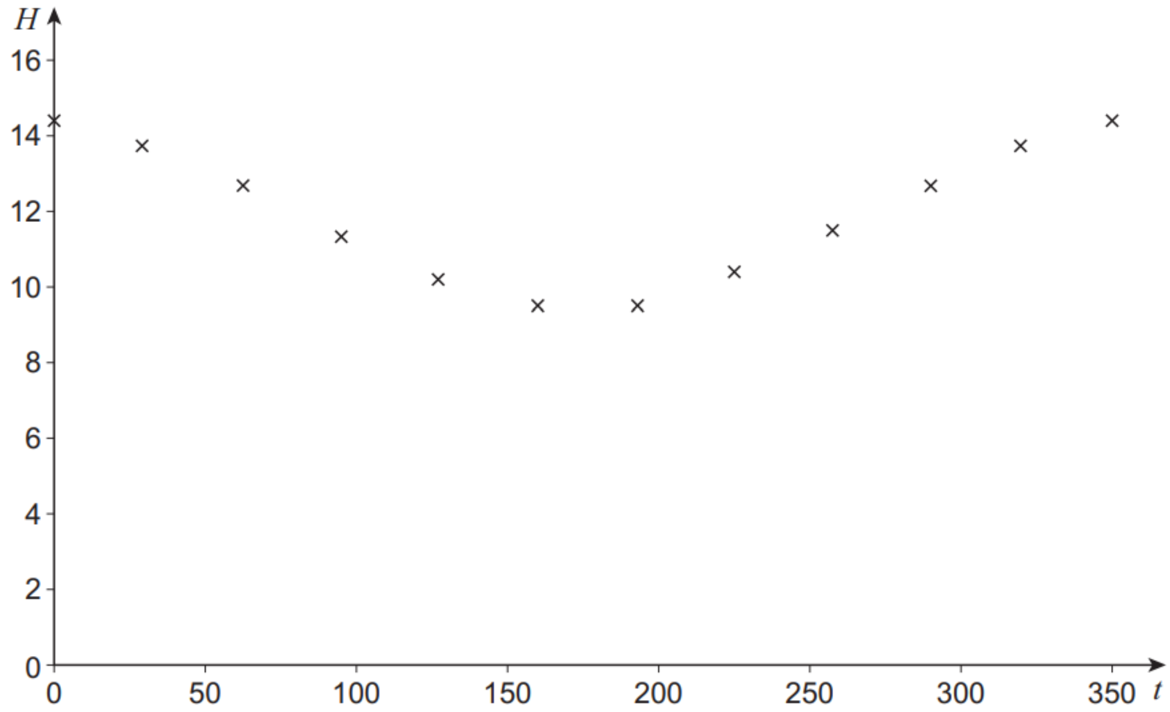
$= \boxed{109 \text{ days}}$



8 (c) Mike's friend Sofia, who lives in Spain, also records the number of hours of darkness on various days throughout the year.

Her results are shown in **Figure 2** below.

Figure 2



Sofia attempts to model her data by refining Mike's model.

She decides to increase the 3.87 value, leaving everything else unchanged.

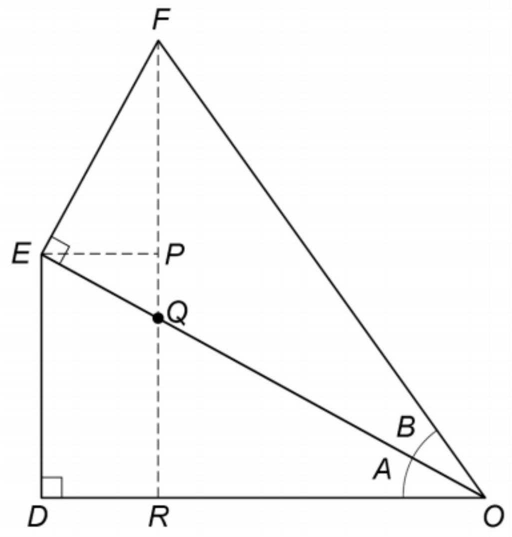
Explain whether Sofia's refinement is appropriate.

[2 marks]

- increasing 3.87 increase the amplitude of curve (vertical stretch)
- her curve has a smaller amplitude than Mike's so increasing 3.87 is not appropriate

14 Some students are trying to prove an identity for $\sin(A+B)$.

They start by drawing two right-angled triangles *ODE* and *OEF*, as shown.



The students' incomplete proof continues,

Let angle $DOE = A$ and angle $EOF = B$.

In triangle *OFR*,

Line 1 $\sin(A+B) = \dfrac{RF}{OF}$

Line 2 $= \dfrac{RP+PF}{OF}$

Line 3 $= \dfrac{DE}{OF} + \dfrac{PF}{OF}$ since $DE = RP$

Line 4 $= \dfrac{DE}{.....} \times \dfrac{.....}{OF} + \dfrac{PF}{EF} \times \dfrac{EF}{OF}$

Line 5 $= \text{..............} + \cos A \sin B$

14 (a) Explain why $\dfrac{PF}{EF} \times \dfrac{EF}{OF}$ in Line 4 leads to $\cos A \sin B$ in Line 5

[2 marks]

$\angle OQR = \angle FQE$ vertically opposite angles

$\angle ORQ = \angle FEQ = 90°$

so $\angle EFQ = A = \angle QFE$ or $\angle PFE$

$\boxed{\dfrac{PF}{EF} = \cos A \quad \dfrac{EF}{OF} = \sin B \quad \text{in triangle } CEF}$

14 (b) Complete Line 4 and Line 5 to prove the identity

Line 4 $$= \frac{DE}{OE} \times \frac{OE}{OF} + \frac{PF}{EF} \times \frac{EF}{OF}$$

Line 5 $= \ldots \sin A \cos B \ldots + \cos A \sin B$

[1 mark]

14 (c) Explain why the argument used in part **(a)** only proves the identity when A and B are acute angles.

[1 mark]

- single proof based on diagram which uses right-angled triangles, assumed that A & B are acute
- proof only for acute an les

14 (d) Another student claims that by replacing B with $-B$ in the identity for $\sin(A+B)$ it is possible to find an identity for $\sin(A-B)$.

Assuming the identity for $\sin(A+B)$ is correct for all values of A and B, prove a similar result for $\sin(A-B)$.

[3 marks]

$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$

$\sin(-B) = -\sin B$

$\cos(-B) = \cos B$

$$\boxed{\begin{aligned}&\sin(A-B)\\ &= \sin A \cos B - \cos A \sin B\end{aligned}}$$